Eigen value of a Matrix by power method.

Introduction :

For every square matrix A, there is a scalar λ and a non-zero column vector X such that $AX = \lambda X$. Then the scalar λ is called an eigen value of A and X, the corresponding eigen vector.

Power method is used to determine numerically largest eigen value and the corresponding eigen vector of a matrix A.

Smallest Eigen values of a square matrix :

By property of eigen values and eigen vectors if λ is an eigen value of A and X is the corresponding eigen vector, then $\frac{1}{\lambda}$ is an eigen value of A^{-1} with the same eigen vector X.

The smallest eigen value of a square matrix A can be found as follows. First compute the inverse matrix B. Then power method is applied to the inverse matrix B, which gives the largest eigen value λ of B and the corresponding eigen vector X.

Hence the smallest eigen value of A $= \frac{1}{\lambda}$ and the corresponding eigen vector is X.

Computation of all Eigen values of a square matrix :

Let A be a given 3×3 square matrix. First find the largest eigen value λ_1 of A by using power method. Then consider the matrix $B = A - \lambda_1 I$. Again power method is applied to find the dominant eigen value of B. Then the smallest eigen value of A is equal to the dominant eigen value of $B + \lambda_1$.

The third eigen value of the matrix A is found by using the property that

Sum of the eigen values of A = Sum of the principal diagonal elements of A.

PROBLEMS :

① Find numerically the largest eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$

Solution :

We choose the initial vector $X_0 = (1, 0, 0)^T$.

Then

$$AX_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$\therefore$ The largest eigen value is 25.18 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$.

(2) Find the dominant Eigen value and the corresponding Eigen vector of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find also the other two eigen values.

Solution :

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an initial vector.

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$$AX_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.429 \\ 0 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.429 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.574 \\ 1.858 \\ 0 \end{bmatrix} = 3.574 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.97 \\ 1.99 \\ 0 \end{bmatrix} = 3.97 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

The largest eigen value of $A = 4$ and its eigen vector is $\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$.

The least eigen value of A is the largest eigen value of $B = A - 4I$.

$$B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By power method, the dominant eigen value of B is obtained as follows.

Let $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be the initial vector.

Then

$$BX_0 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}$$

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$$BX_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}$$

$$BX_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigen value of $B = -5$

$\therefore$ The smallest eigen value of $A = B + 4I$

$= -5 + 4 = -1$

By property,

Sum of the eigen values $=$ Trace of A

$= 1 + 2 + 3 = 6$.

$\therefore$ If λ_3 is the third eigen value,

$4 - 1 + \lambda_3 = 6$

$\Rightarrow \lambda_3 = 6 - 3 = 3$.

$\therefore$ The eigen values are $4, 3, -1$.

(3) Find the dominant eigen value of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ by power method.

Solution :

Let $X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$AX_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 0.43 \\ 1 \end{bmatrix} = 7X_2$$

$$AX_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.43 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.43 \\ 5.29 \end{bmatrix} = 5.29\begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = 5.29X_3$$

$$AX_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 5.38 \end{bmatrix} = 5.38\begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = 5.38X_4$$

$$AX_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 5.38 \end{bmatrix} = 5.38\begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = 5.38X_5$$

$\therefore$ The dominant eigen value $\lambda = 5.38$

and the corresponding eigen vector $= \begin{pmatrix} 0.46 \\ 1 \end{pmatrix}$.

Homework problems :

1. Find the power method, the largest eigen value of $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ correct to two decimal places, choose $(1,1)^T$ as the initial eigen vector.

Ans: The eigenvalue $\lambda = 4.62$

eigen vector $= \begin{pmatrix} 1 \\ 0.62 \end{pmatrix}$.

(2). Find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Ans:

Eigen value $\lambda = 3.414$

Eigen vector $x = (0.707, -1, 0.707)^T$.

(3). Determine the dominant eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ using power method.

Ans:

The largest eigen value $\lambda = 20.124$

Eigen vector $x = \begin{pmatrix} 0.062 \\ 1 \\ 0.062 \end{pmatrix}$